Final study guide

The final will cover everything before the midterm (see midterm study guide) and after, with an emphasis (Think 75%) on the material not covered on the midterm, which includes the following topics:

Quotient groups + Normal subgroups

- Def of a quotient group in terms of a homomorphism $\varphi: G \to H$. (Fibers are the elements)
- equivalent definitions of normality: $N \trianglelefteq G \iff xNx^{-1} = N \;\forall x \in G$

  $\iff xN = Nx \;\forall x \in G \iff N_G(N) = G \iff$ coset multiplication is well-defined $\iff N$ is the kernel of some homomorphism $G \to H$.
- left cosets of a subgroup $H \leq G$: $aH = bH \iff b \in aH \iff b^{-1}a \in H$.
- cosets of H partition G.
- If $N \trianglelefteq G$, definition of G/N. What are the elements? How do you multiply them? $(aNbN = abN)$
- Lagrange's Theorem (G finite, $H \leq G \Rightarrow \frac{|G|}{|H|} = |G:H|$)
- Cauchy's Theorem (G finite abelian, p a prime dividing $|G|$ $\Rightarrow G$ has an element of order p)

Isomorphism Thms

- The first is the most important: $\varphi : G \to H$ a homomorphism $\Rightarrow G/\ker\varphi \cong \operatorname{im}\varphi$. (if $K = \ker\varphi$, the isomorphism $G/K \to \operatorname{im}\varphi$ is defined $aK \mapsto \varphi(a)$. Do you see why this is well-defined?)

- Understand the statements and proofs of the other iso theorems, but you don't need to memorize them.

The alternating group

- Every elt of S_n has a sign: odd or even. How do you determine the sign of an element?

- the subgroup of even permutations of S_n is A_n. $A_n \trianglelefteq S_n$.

Group actions

- If G acts on A, and $g \in G$, then $\sigma_g \in S_A$, defined $\sigma_g(a) = g \cdot a$.

- The map $G \to S_A$ defined $g \mapsto \sigma_g$ is a homomorphism, called the permutation representation of the action

- kernel of an action; action is faithful if $\ker = 1$. What does a faithful action look like?

- orbits: What are the orbits of an action? Action is transitive $\iff$ exactly one orbit. What does a transitive action look like?

- $G_a :=$ stabilizer of a in $G = g \in G$ s.t. $g \cdot a = a$. $G_a \leq G$.

- $|G : G_a| =$ # of elements in the orbit of a

Groups acting by left multiplication

- G acts on itself by $g \cdot a = ga$; it's faithful + transitive.

- Get a homomorphism $G \to S_G$ that is injective (Cayley's Thm), so $G \cong$ image of $G \to S_G$

- G acts on left cosets of $H \leq G$ by $g \cdot aH = gaH$

- if A = set of left cosets of H, then we get a homomorphism $G \to S_A$

- If p is the smallest prime dividing $|G|$, then any subgroup of index p is normal

Groups acting by conjugation

- G acts on itself by conjugation: $g \cdot a = gag^{-1}$.

- a is conjugate to $b \Leftrightarrow gag^{-1} = b$, some $g \in G \Leftrightarrow a$ and b are in the same orbit under conjugation action (orbits in this case are called conjugacy classes)

- # of conjugates of $a \in G = |G : C_G(a)|$

- $\{a\}$ is a conj. class $\Leftrightarrow a \in Z(G)$.

- class equation: $g_1, \dots, g_r$ representatives of conj. classes not in the center $\Rightarrow$ $|G| = |Z(G)| + \sum_{i=1}^{r} |G : C_G(g_i)|$.

- Group of order p^α $(\alpha \geq 1)$ has nontrivial center.

- two elements of S_n are conjugate $\Longleftrightarrow$ their cycle decomp has the same # of cycles of each length.
  # of conj. classes of S_n = # of partitions of n

- $\tau (a_1\, a_2 \dots a_r) \tau^{-1} = (\tau(a_1)\, \tau(a_2) \dots \tau(a_r))$

- Subtlety: if $\sigma, \tau \in A_n$, then if σ and τ are conj. in A_n, they are conj. in S_n, but not vice versa.

  e.g. in S_3, $(1\,2\,3)$ is conj. to $(1\,3\,2)$ since $(2\,3)(1\,2\,3)(2\,3)^{-1} = (1\,3\,2)$, but $(23) \notin A_n$, so they're not conj. in A_n.